A Geometric Approach to Cylindrical Algebraic Decomposition

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The Speaker

Speaker. Rizeng Chen

- I am Rizeng Chen (pronounced as "ReeTseng Chern").
- I am currently a fifth-year PhD candidate from Peking University, China.
- My advisor is Prof. Bican Xia.
- My research interest lies in the intersection of algebraic geometry, logic, and theoretical computer science.

Question

Question: Have you ever dreamed of someone who proves everything for you?

Leibniz's Dream

Leibniz had dreamed so, hence he invented the first calculator that can perform all four basic arithmetic operations.

Figure 1: Leibniz's monument in Leipzig (Credit: Ad Meskens)

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- Hilbert and Ackermann called it "das Hauptproblem der mathematischen Logik" – "the main problem of mathematical logic".

Decision Problem over the Reals

We are interested in the decision problem over the reals.

The Decision Problem

- In the first order theory of the reals, an atom is a polynomial equation $f = 0$ or a polynomial inequality $f > 0$.
- Using logical connectives $(\wedge, \vee, \neg)$, atoms can be combined to get a quantifier-free formula $\Psi(x_1, \ldots, x_n)$.
- A closed formula is a fully quantified formula

$$\Phi = (Q_1 x_1)(Q_2 x_2) \cdots (Q_n x_n)\Psi(x_1, \ldots, x_n),$$

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E.g., for all a, b, c, there exists x such that $x^3 + ax^2 + bx + c = 0$ holds.

$$(\forall a)(\forall b)(\forall c)(\exists x)(x^3 + ax^2 + bx + c = 0).$$

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- Algebra (Real Nullstellensatz, Positivstellensatz)
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- Computation
  Certainly we need an algorithm!

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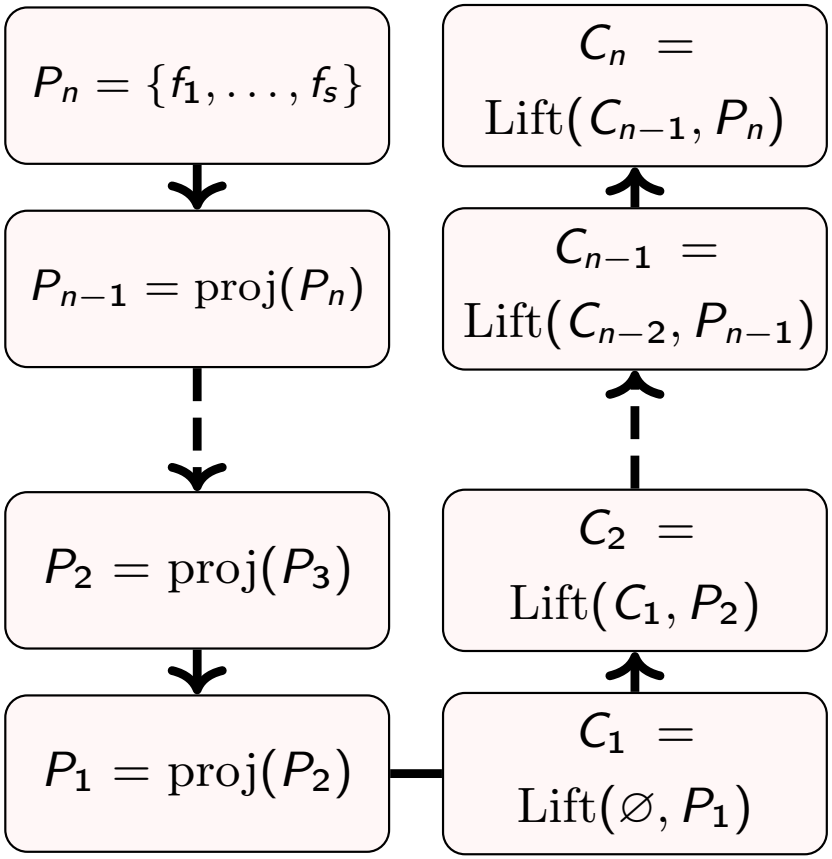
- Tarski's result has a non-elementary complexity, which makes it useless in practical computation.
- Collins (1975) proposed Cylindrical Algebraic Decomposition (CAD), which has been shown to be the most efficient decision algorithm so far.

A Glimpse of Cylindrical Algebraic Decomposition



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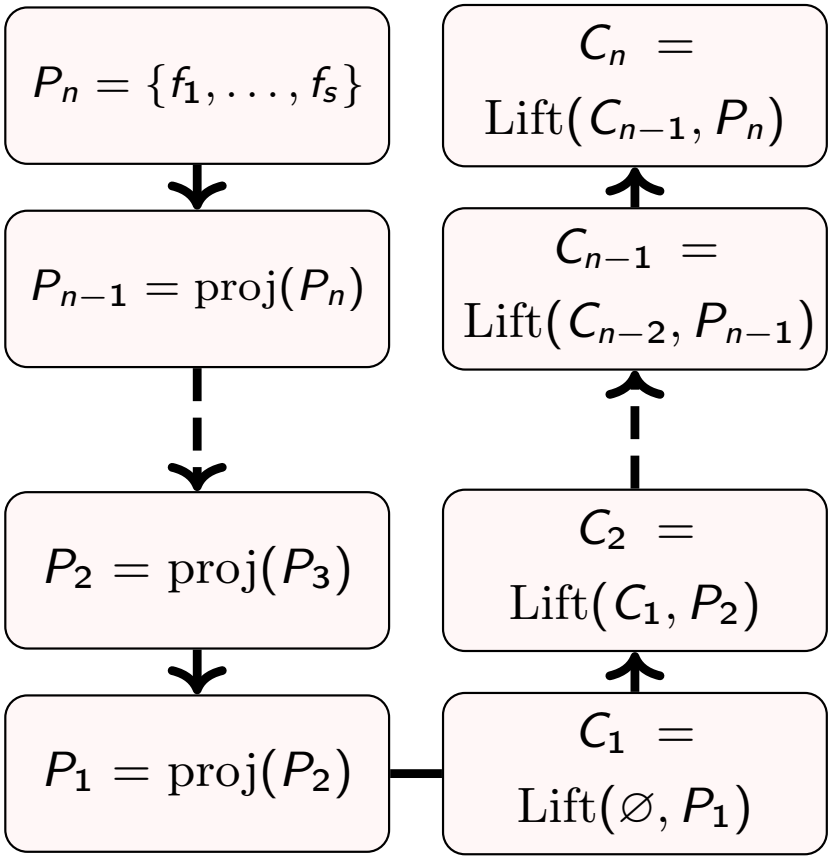

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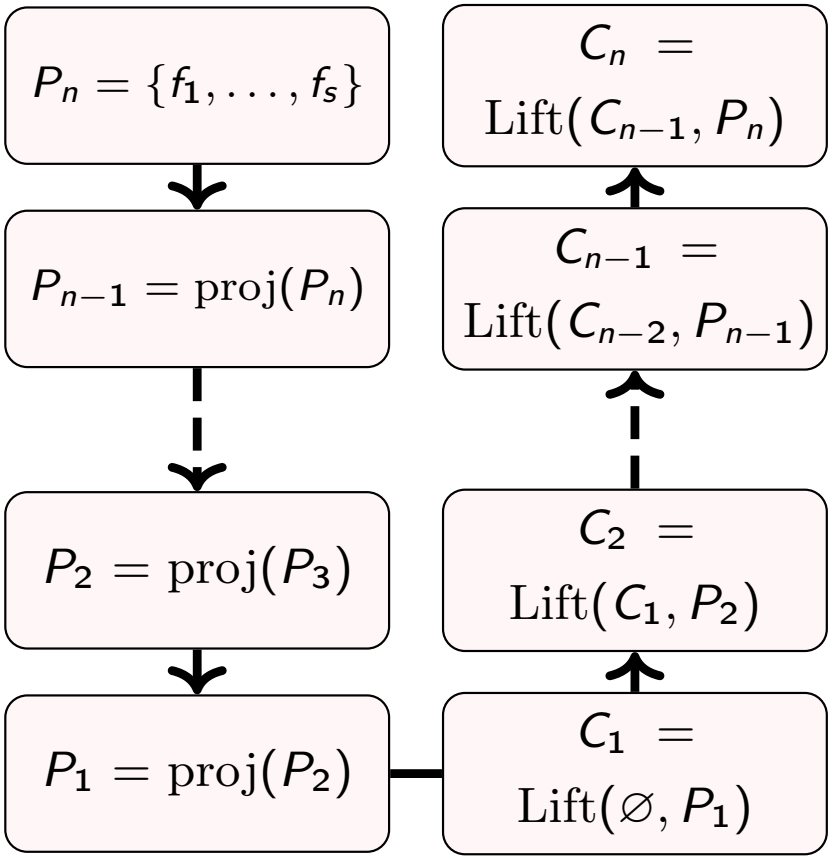

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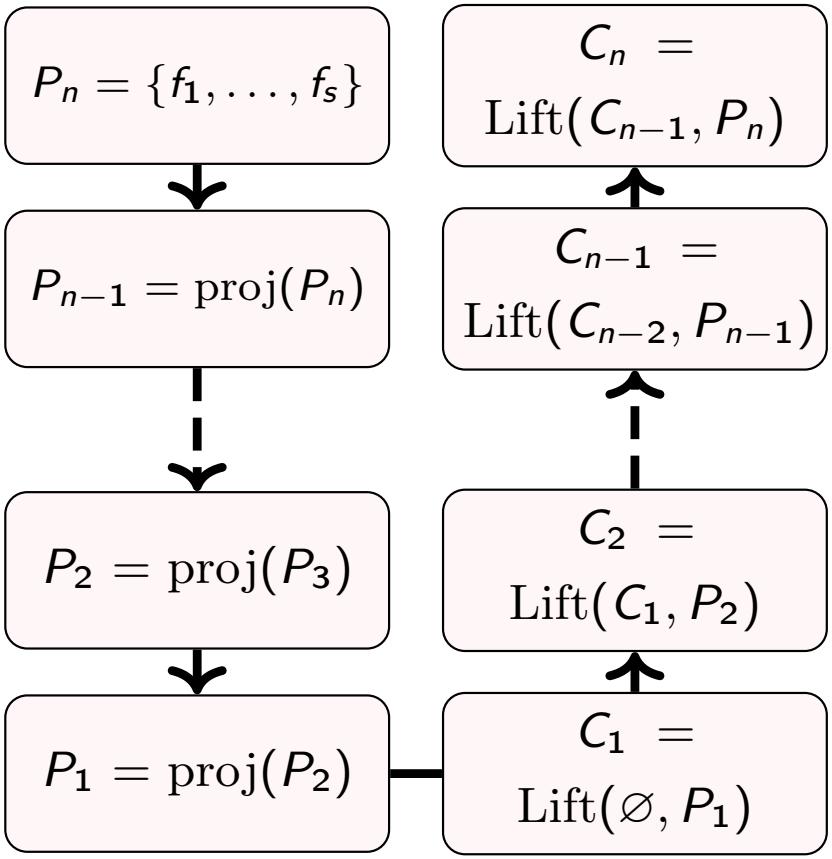

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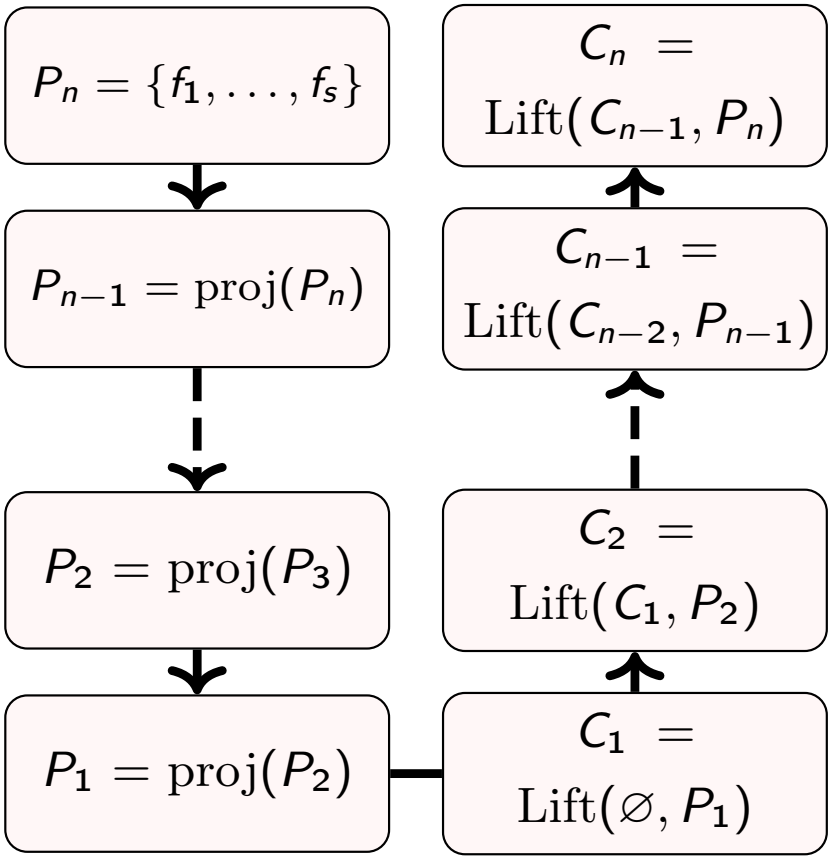

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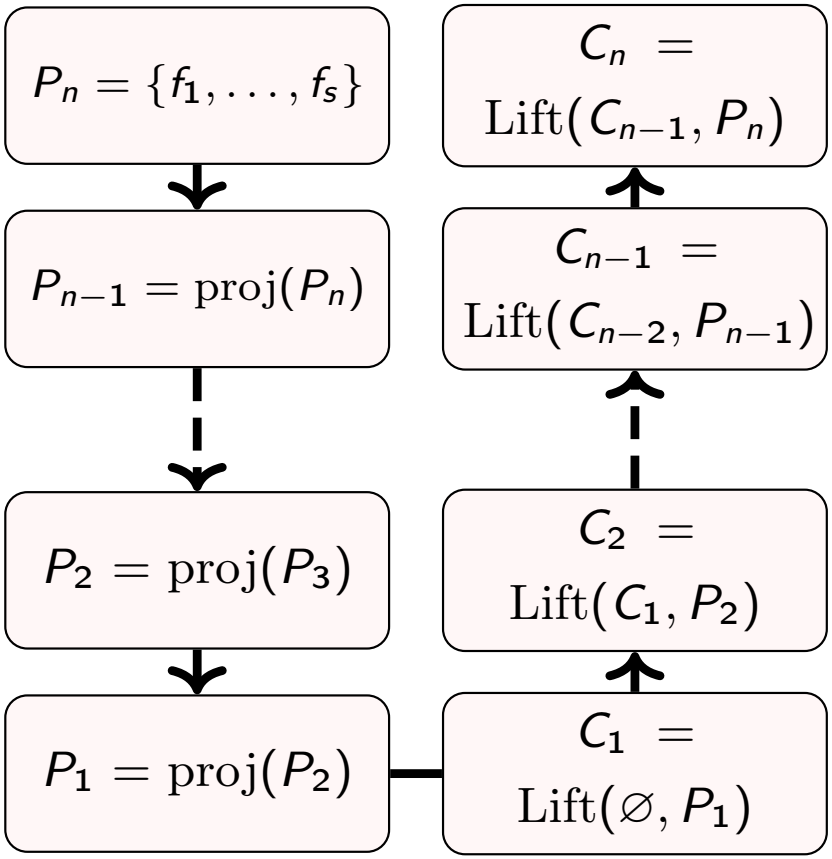

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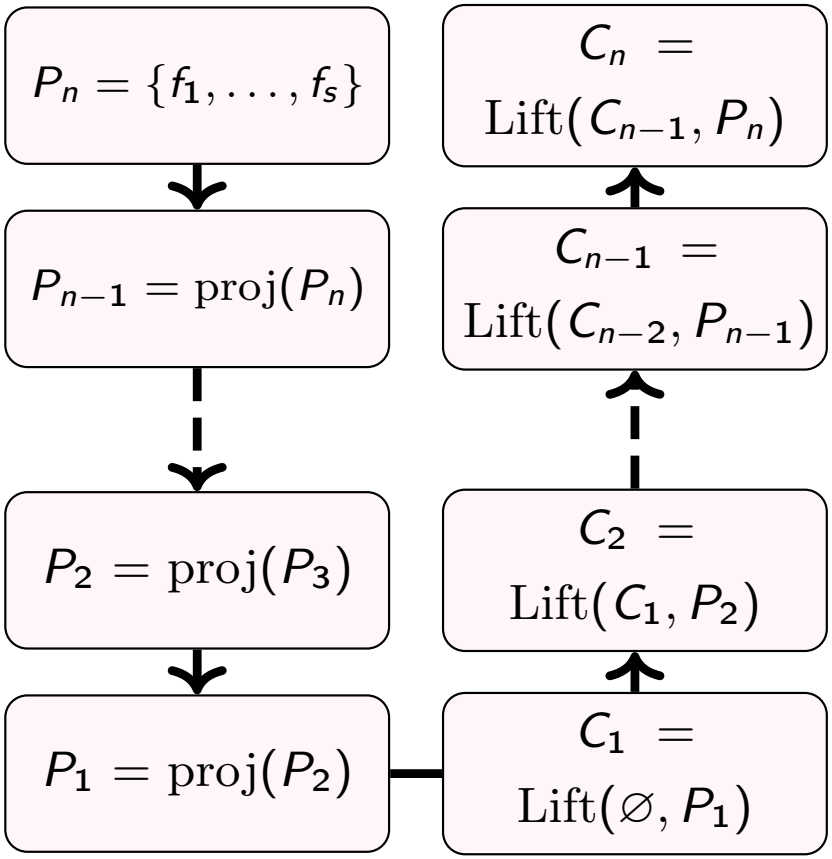

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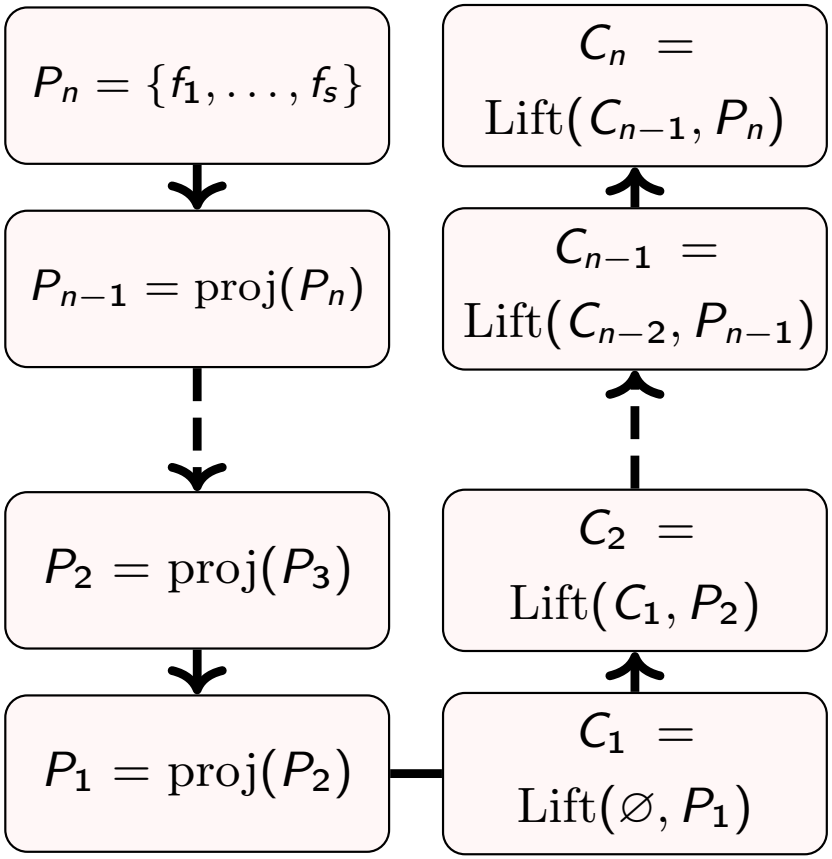

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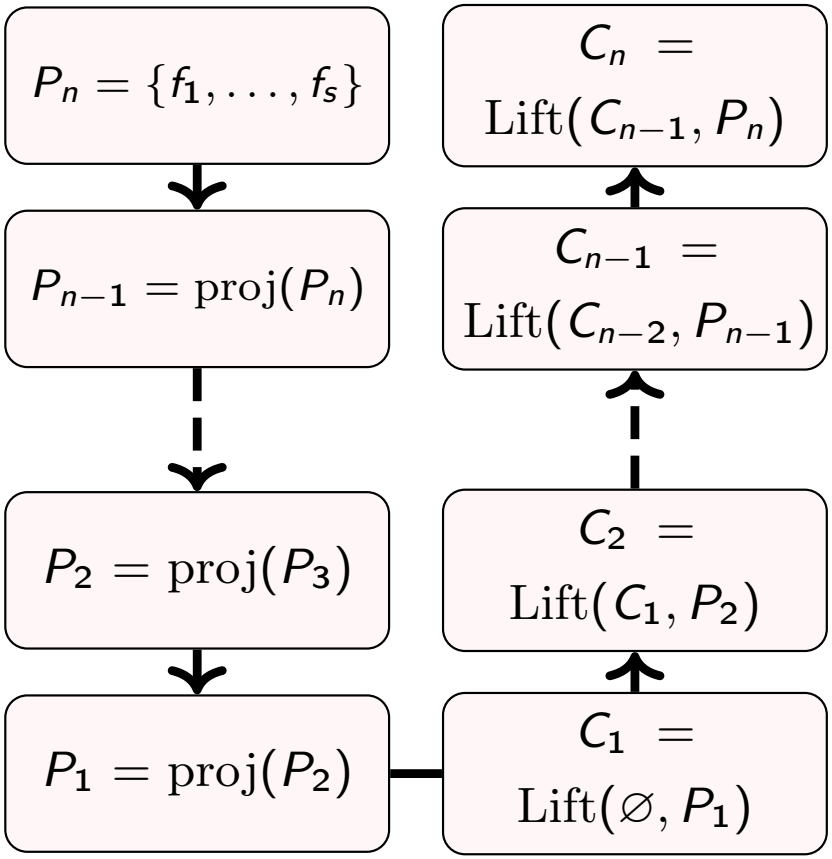

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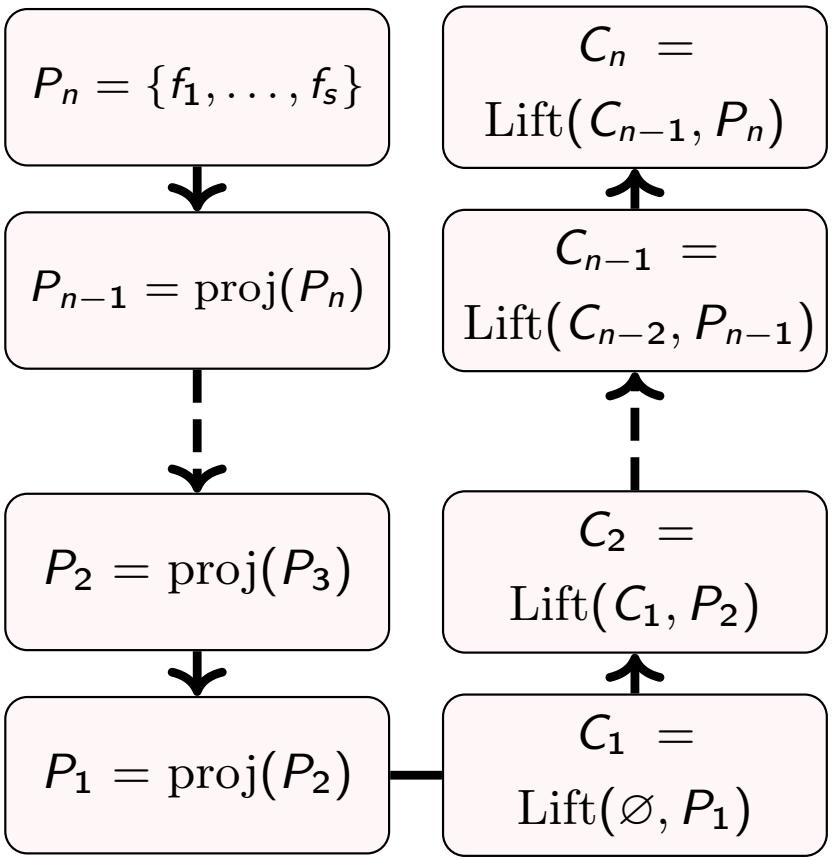

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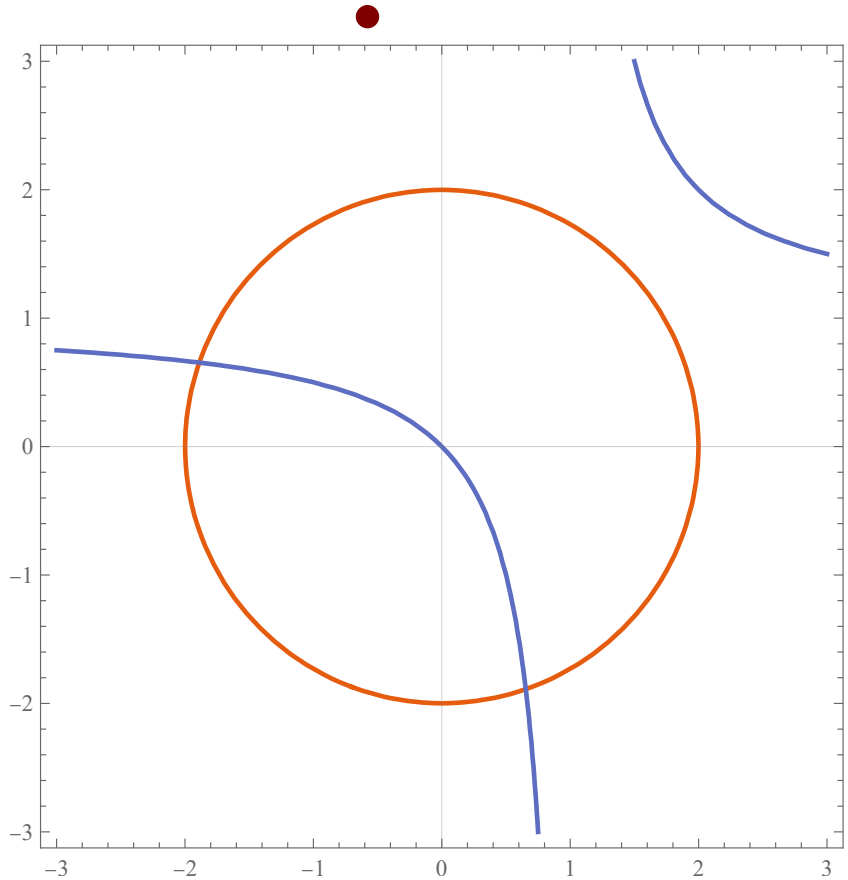

Figure 3: An Example Input

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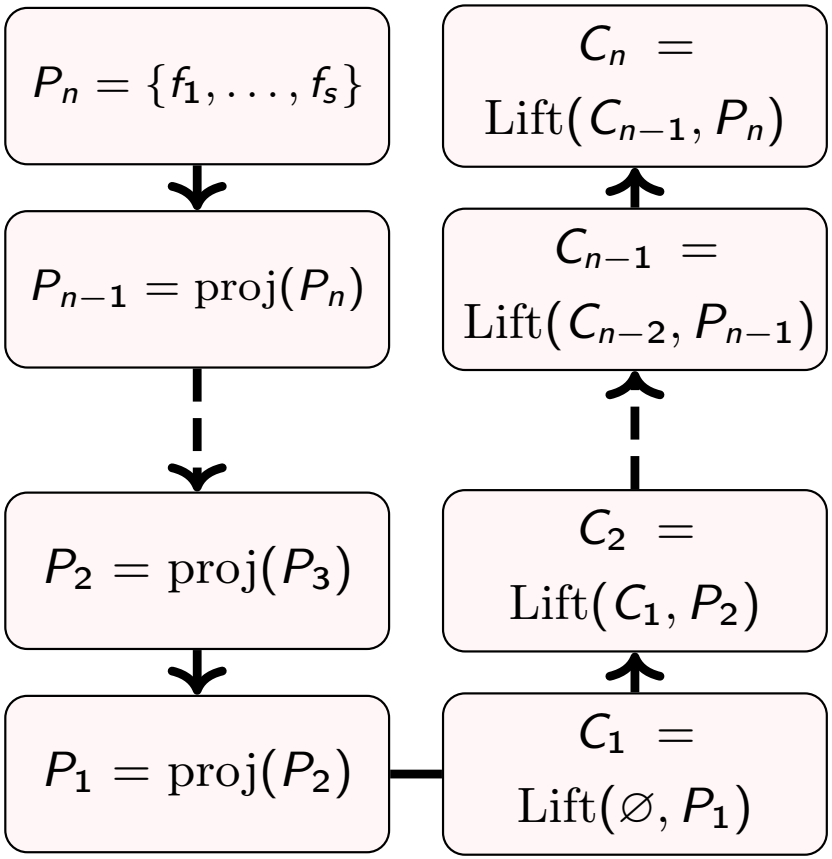

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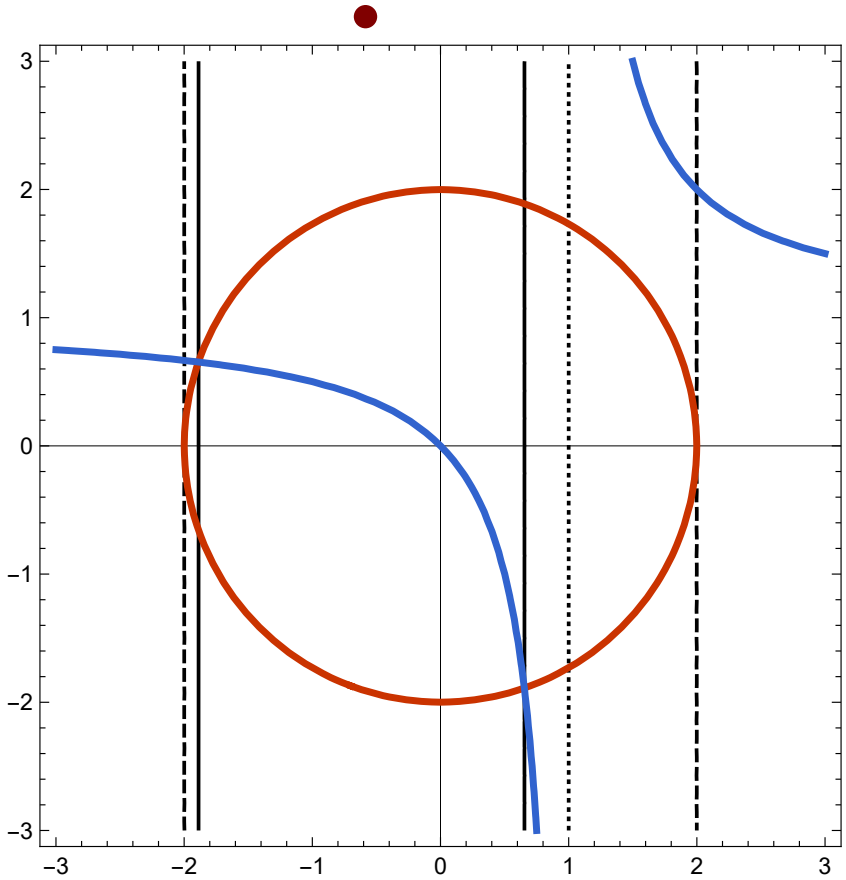

Figure 3: Projection Phase

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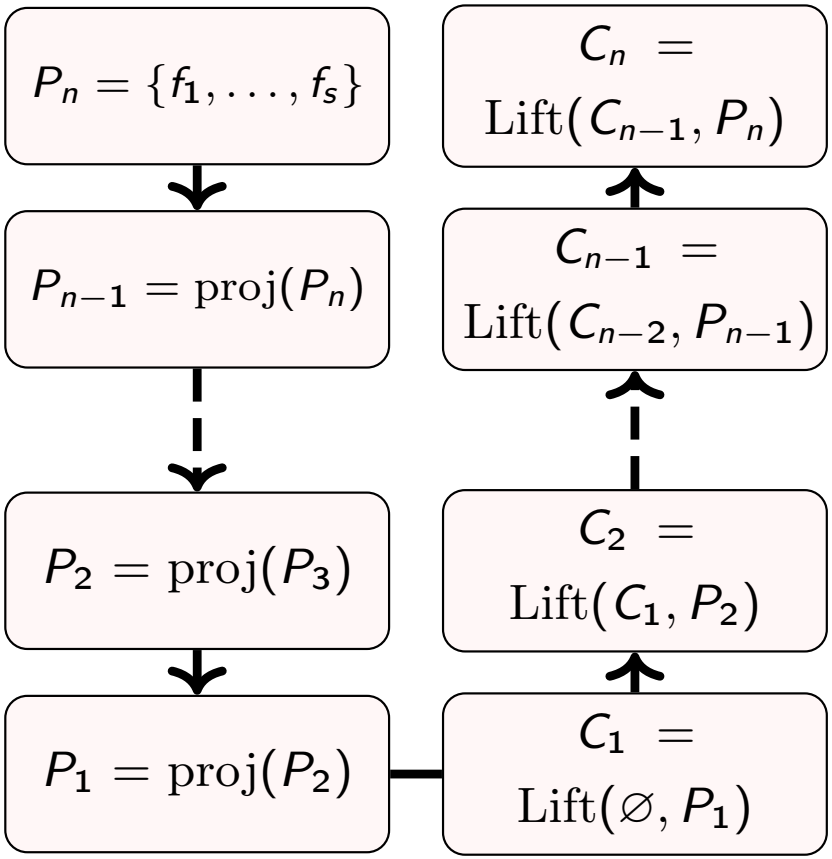

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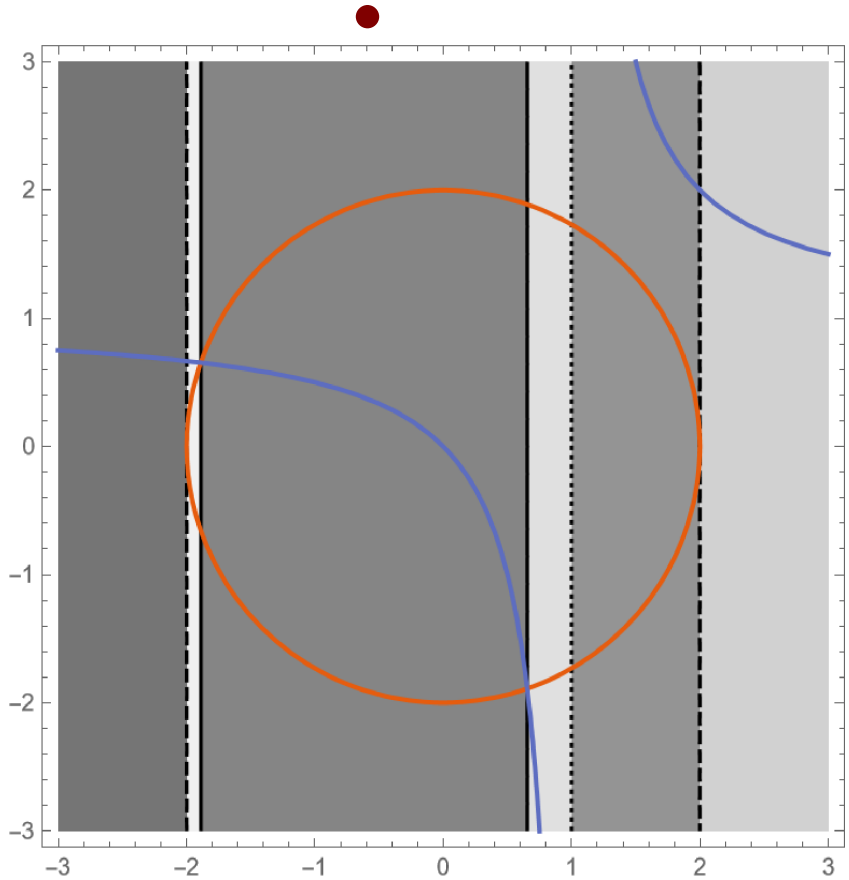

Figure 3: P_1-sign-invariant regions

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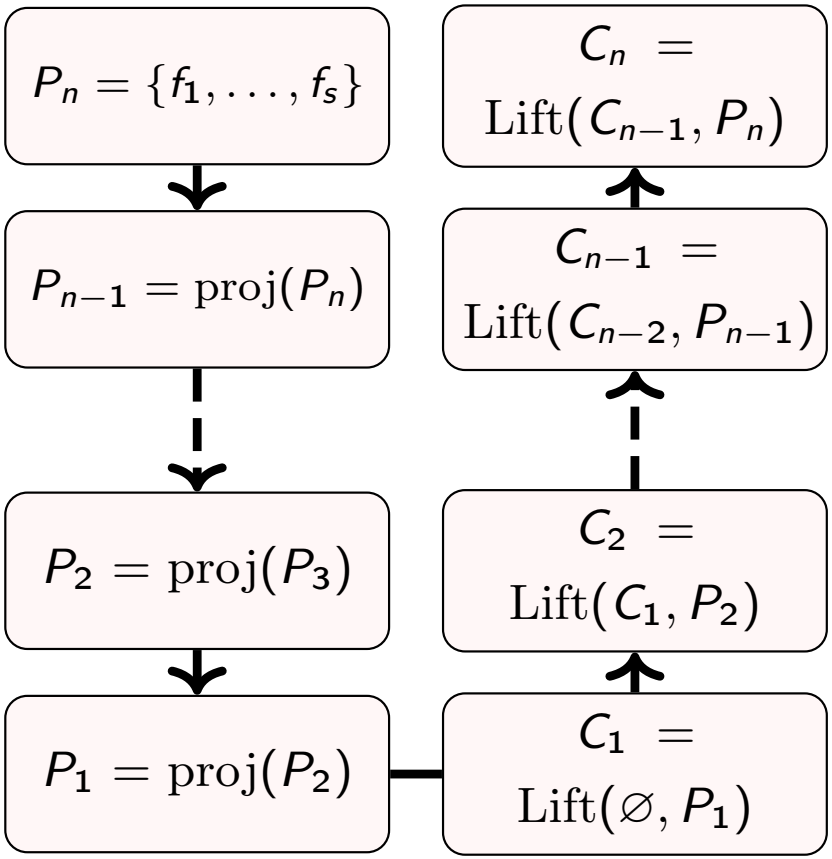

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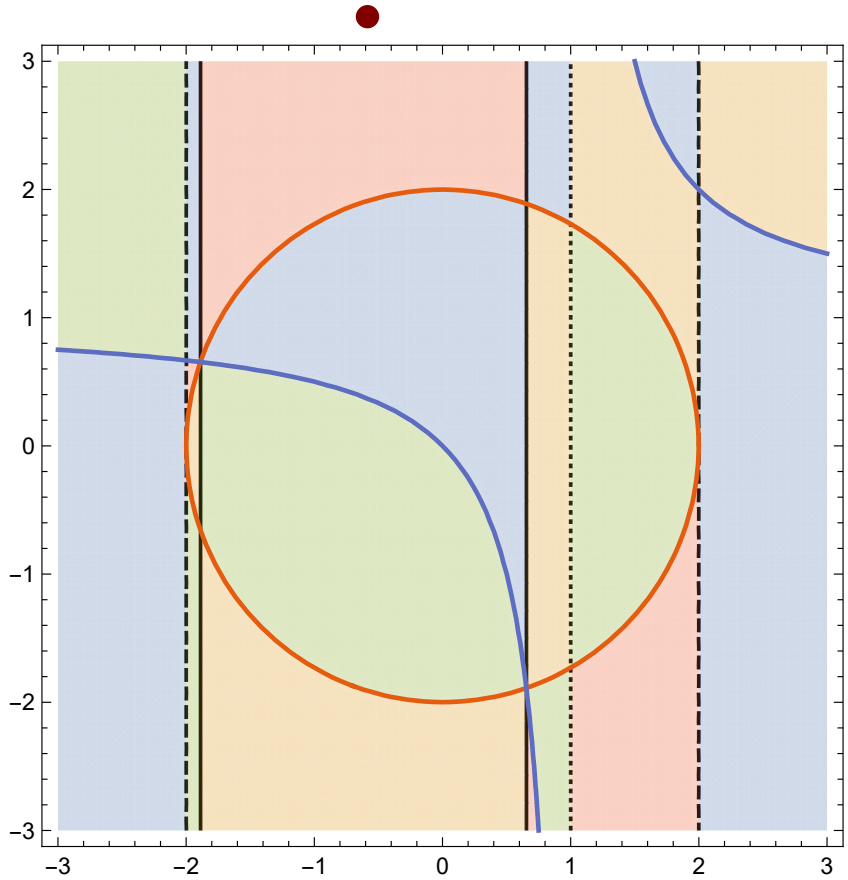

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- "Leading coefficient or discriminant eliminates a variable." ?????

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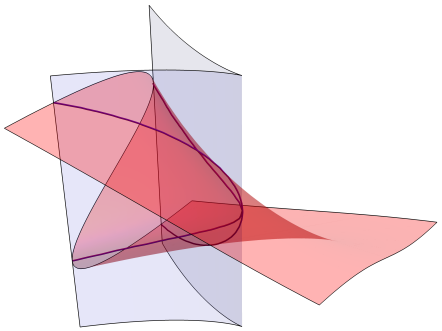
- For some applications, there are already equations in the input (e.g. Real Root Classification, Automatic Theorem Proving).
- Also, equations are generated during the projection phase.
- To see this, let us consider a toy polynomial system

$$x^3 + px + q = 4p^3 + 27q^2 = 0$$

in parameters p, q and variable x.

The Challenge from Equations



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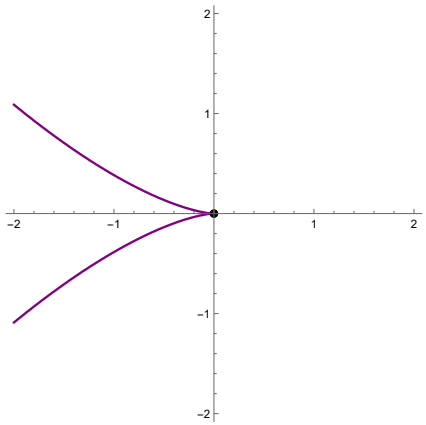

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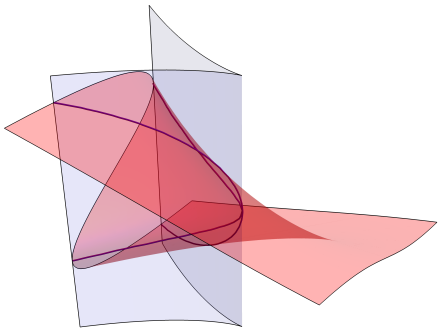

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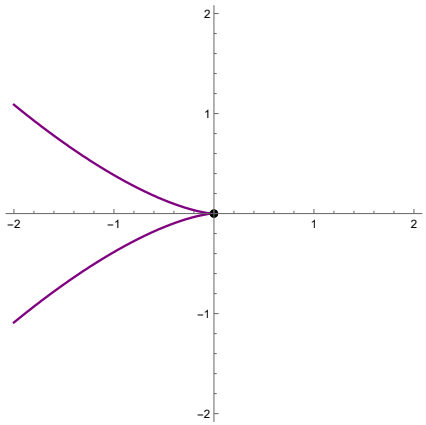

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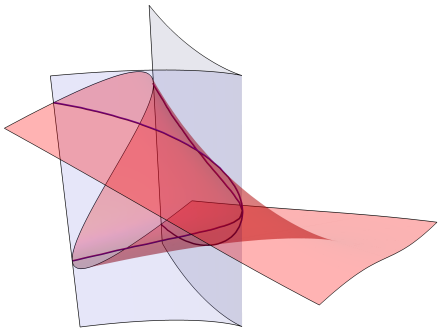

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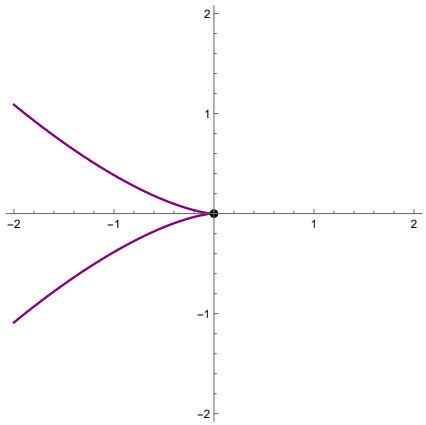

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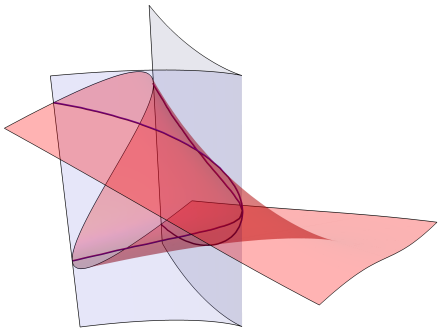

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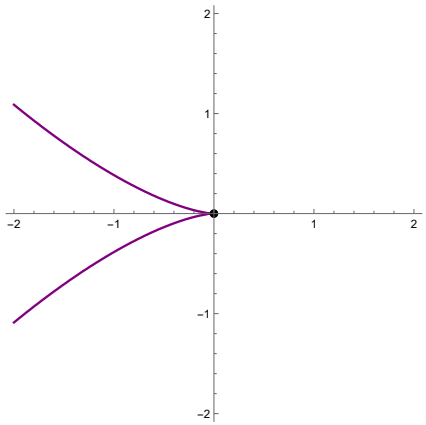

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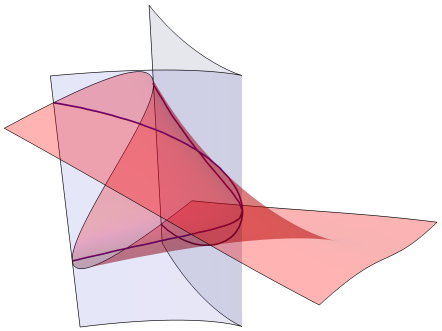

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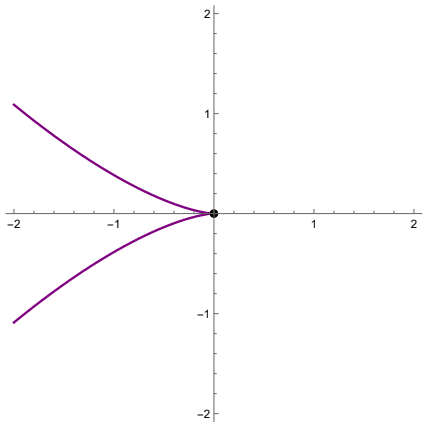

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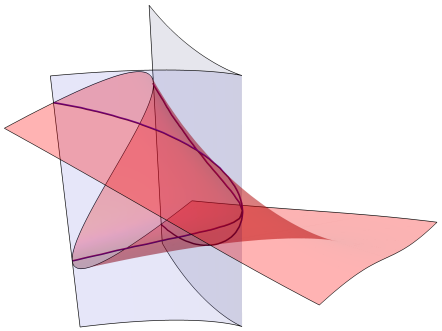

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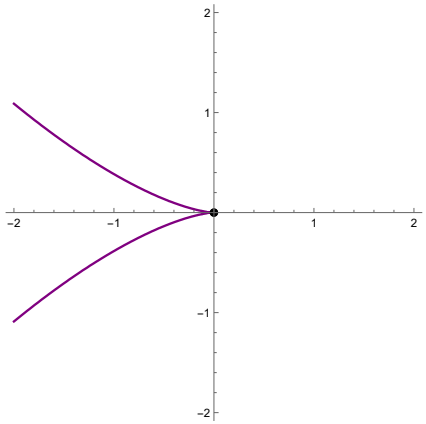
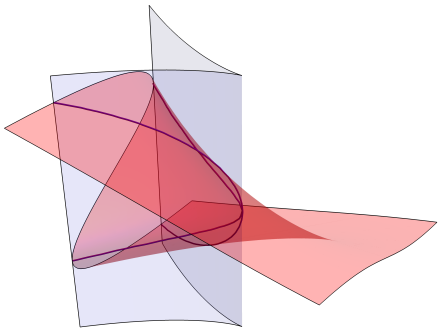


(b) The projection

Figure 3: Equations arise

- For larger non-trivial inputs, this phenomenon becomes much more frequent.
- There are already many equations in the input and many additional equations are systematically generated during the projection phase.

The Challenge from Equations



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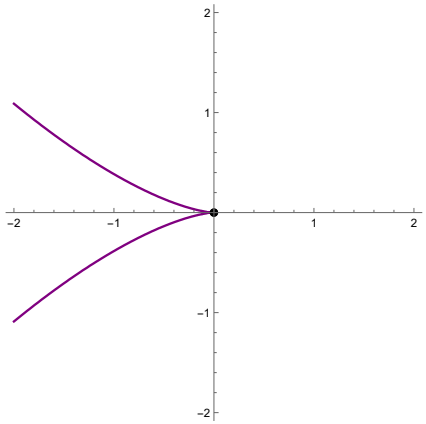

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Figure 3: Equations arise

- For larger non-trivial inputs, this phenomenon becomes much more frequent.
- There are already many equations in the input and many additional equations are systematically generated during the projection phase.
- Therefore it is an important challenge to exploit these equations.

Previous Results

There has been intensive efforts regarding this issue.

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- England, Bradford and Davenport (2015) found that savings can be made during the lifting phase.

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  - Its correctness is conditional only.
  - When the projection is not quasi-finite (a rare case), their algorithms would raise an error (the "nullification problem").

A new simple algorithm

We develop a new conceptually simple and efficient algorithm that exploits all the equations systematically in the recent paper

R. Chen, *A geometric approach to cylindrical algebraic decomposition*, Math. Comp. **95** (2026), pp. 2025-2059.

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- Non-example: $ax^2 + bx + c = 0$. Because a might be 0, $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is not continuous near $a = 0$.

The big idea

- We do not "project" polynomials anymore. We will project varieties.
- Very roughly speaking, our algorithm starts with some varieties.
- Then we stratify their projection images into
  - "flat" ($\approx$ fibers changes continuously) and
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- Non-example: $x^2-t=0$. Generically there are two distinct solutions, except when $t=0$.

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- For a finite, flat morphism $\pi : X \to Y$ with $\Omega_{X/Y} = 0$, $n(y)$ is locally constant.
- Conversely, one can show that for a quasi-finite flat morphism $\pi : X \to Y$ with locally constant $n(y)$, the reduced map $X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is finite, étale, hence $\Omega_{X_{\mathrm{red}}/Y_{\mathrm{red}}} = 0$.

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Folklore. Effective Generic Freeness

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- Applying effective generic freeness recursively, we can stratify the projection image into "flat" pieces.

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- Then on each piece, a trace pairing technique (Hermite's Quadratic Form) can be used to count the complex solutions, so each piece is further stratified into "unramified" pieces.

Hermite's Quadratic Form

- Suppose $A = k[x_1, \ldots, x_n]/I$ is a zero-dimensional k-algebra ($\operatorname{char} k = 0$). The Hermite's Quadratic Form is the trace pairing

$$\begin{array}{rcl} H : A \times A & \to & k \\ (f, g) & \mapsto & \operatorname{tr} L_{fg} \end{array},$$

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  where L_{fg} is the k-linear multiplication map $A \to A, a \mapsto fg \cdot a$.
- $\operatorname{rank} H = \# V_C(I)$, $\operatorname{sign} H = \# V_R(I)$.
- This can be generalized to the parametric case. If $\operatorname{rank} H$ is a constant, regardless of the parameters, then the system is "unramified".

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- Now we can conclude the existence of semi-algebraic sections on (connected components) of each stratum.
- So a c.a.d. adapted to X can be lifted from a c.a.d. adapted to the strata.

Foundation Rebuilt

Theorem (Chen, 2026)

Suppose that

$$\pi : X \underset{\text{closed}}{\hookrightarrow} \mathbb{A}^1_Y \to Y$$

is finite free morphism between affine R-varieties.
If $S \subseteq Y(R)$ is a semi-algebraically connected semi-algebraic set on which $\#X_{\overline{y}}$ is a constant, then there are semi-algebraic continuous functions

$$\xi_1, \ldots, \xi_l : S \to X(R)$$

such that $\pi \circ \xi_i = \mathrm{id}_S$ and $\{\xi_i(y)\}_{i=1}^l$ is the real fiber of y in $X(R)$.

This is a non-trivial generalization of Collins (1975) in the algebro-geometric context.

The Meaning

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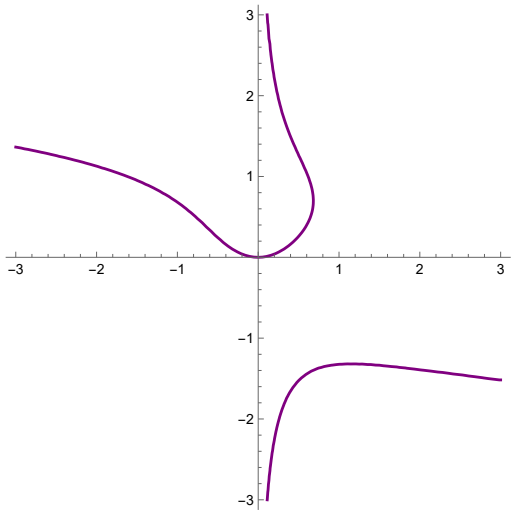
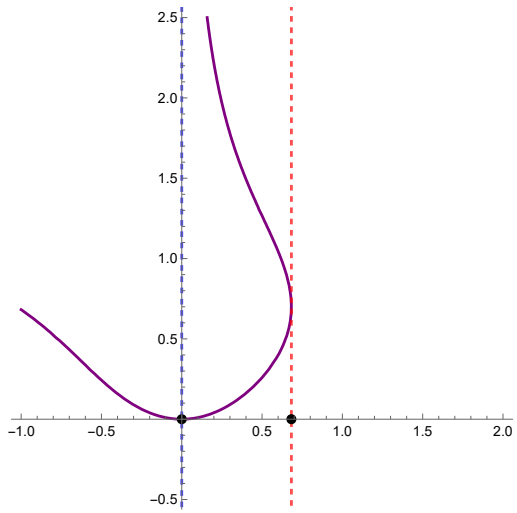
- This is non-trivial because it links the number of complex solutions (geometric invariant) to being a covering on the real points (semi-algebraic property).
- Therefore, it bridges complex algebraic geometry and real algebraic geometry.
- The shortest path between two truths in the real domain passes through the complex domain.

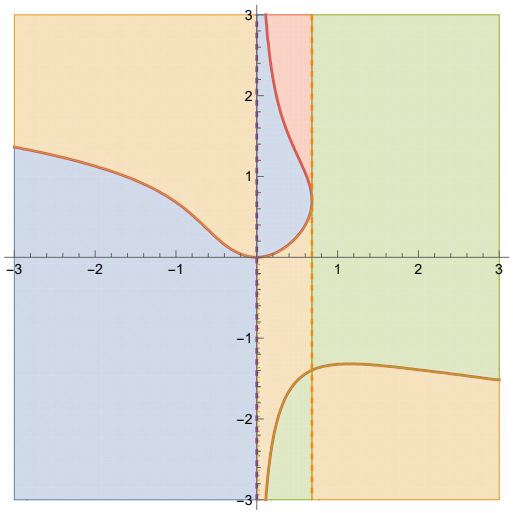
Illustration



(a) Locus of $f = xy^3 - y + x^2$



(b) Where "flatness" or "unramifiedness" fails



(c) The final c.a.d.

Figure 4: An example of c.a.d.

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So Collins-Hong projection operator contains all the polynomials (possibly more than) needed to encode a stratification of the projection from a hypersurface $V(f)$ into "flat" and "unramified" pieces.
Therefore proj is not really eliminating variables, but stratifying the projection image. :)

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- Using our geometric method, this problem is reduced to compute a lower-dimensional c.a.d. adapted to another family of varieties recursively.
  - Our algorithm generates a c.a.d. consisting of **75** cells.

Comparison

To see where the gain comes from, we need to look deeper into the projection phase.

The left column shows the projection polynomials computed by classical methods, while varieties generated by our geometric method are shown in the right column.

	classical	geometric
Level 5	$w^2 + (x+y)^2 u - yz$	$V(0), V(f)$
Level 4	$(x+y)^2 u - yz$	$V(0), V((x+y)^2 u - yz)$
Level 3	$(x+y)^2, yz$	$V(0), V(x+y), V(x+y, yz)$
Level 2	$x+y, y$	$V(0), V(x+y), V(x, y)$
Level 1	x	$V(0), V(x)$

Table 1: A comparison of computation of f-sign-invariant c.a.d.

More comparison

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f_2 &= w^2 - 2wy + u^2 - 2uz - x, \\
f_3 &= 32y^2 + 168yz + 40yx - 270y + 8z^2 + 20zx - 390z + 4x^2 - 105x + 450 \\
f_4 &= 240yz - 16yx^2 - 532yx - 4480y - 800z^3 - 1240z^2x - 17720z^2 - 408zx^2 \\
&\quad -6214zx + 25240z - 40x^3 - 550x^2 + 5695x + 1050 \text{ and} \\
f_5 &= 320yzx + 8320yz + 32yx^2 + 264yx - 14840y + 240z^2 + 16zx^2 - 372zx \\
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- Our algorithm (`GeometricCAD`): finished in 10.67 seconds, generating 200 cells,
- Classical algorithm (`QEPCAD`): quit after 14 minutes because of a software failure.

Implementation

We implement our algorithm on Mathematica, the source code is available at

https://github.com/xiaxueqaq/GeometricCADv2.

Experimental Results

Time	Hong-4	Hong-5	Hong-6	Hong-7
Geometric CAD	0.56	2.53	9.77	49.52
RC-CAD	1.19	5.16	20.00	133.86
QEPCAD	1.35	1.35	1.70	>3600

Time	Hong-8	cyclic-4	cyclic-5	cyclic-6
Geometric CAD	270.33	0.03	0.58	5.48
RC-CAD	Error	0.06	4.58	>3600
QEPCAD	>3600	1.56	>3600	>3600

Table 2: Running time in seconds

Experimental Results

Time	FIS-1	FIS-2	FIS-3	FIS-4
Geometric CAD	16.11	2.66	1654.84	0.19
RC-CAD	40.25	1.59	644.91	1.83
QEPCAD	3.68	1101.60	>3600	4.76

Time	FIS-5	FIS-6	FIS-7	FIS-8
Geometric CAD	0.20	2834.00	10.67	12.75
RC-CAD	>3600	>3600	>3600	122.58
QEPCAD	>3600	>3600	ERROR	ERROR

Table 3: Running time in seconds

The CAD Olympiad

	Total	Solved	Gold:Silver:Bronze
Geometric CAD	20	20	15:5:0
RC-CAD	20	14	3:8:3
QEPCAD	20	11	3:1:7

Table 4: Summary

Inside the Discriminant: Multiplicity and Number of Solutions

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- **Theorem (McCallum, 1998)**

  Let $f(y_1, \ldots, y_n, x) \in \mathbb{R}[y_1, \ldots, y_n, x]$, $D(y) = \mathrm{Res}_x(f, \frac{\partial f}{\partial x}) \neq 0$. Let $S \subseteq \mathbb{R}^n$ be a connected submanifold on which f is degree-invariant in x and does not vanish identically, and in which D is multiplicity-invariant. Then f is analytically delineable on S and order-invariant in each f-section over S.

Inside the Discriminant: Multiplicity and Number of Solutions

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- The fewer solutions for the system, the more singular the system is.
- Instead of directly counting the solutions, McCallum (1998) improved CAD by using the multiplicity of discriminant to detect how singular the system is in the hypersurface case.
- **Conjecture**

  Let $\pi : X \to Y$ be a finite flat morphism of reduced k-varieties. The discriminant D is a closed subscheme of Y. Suppose the Hilbert-Samuel multiplicity $\mathrm{mult}_z(D)$ of D is locally constant along a non-singular subscheme Z of Y. Then the geometric fiber cardinality $n(y) = \#X_{\overline{y}}$ is also locally constant along Z.

Inside the Discriminant: Multiplicity and Number of Solutions

Let us work on a concrete example.

- Recall that the discriminant of $x^3 + px + q = 0$ is $4p^3 + 27q^2$.

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- The multiplicity of a smooth point (p, q) on the discriminant is 1, which corresponds to a double root.
- The multiplicity of the cusp $(0, 0)$ is 2, which corresponds to the triple root $x^3 = 0$.

Inside the Discriminant: Multiplicity and Number of Solutions

Dimca and Rosian (1984) showed that for the universal family of monic equations $F(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ of degree n,

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Question: does this pullback to arbitrary finite free corank 1 projection $\pi : X \to Y$ of degree n?

Thank you!

Find more about me on xiaxueqaq.github.io