Deciding first-order formulas involving univariate mixed trigonometric-polynomials

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Authors

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I am Rizeng Chen, the people on the left and I am currently a second-year PhD candidate under the supervision of Prof. Xia.

1. Problem
2. Our Result
3. Implementation
4. Conslusion & Future Work

Introduction - The Intersection Problem

- Consider the parametric curve C_1: $\left\{ \begin{array}{lcl} x(t) & = & t\cos^2 t \\ y(t) & = & t\sin t \end{array} \right.$ and the circle C_2: $x^2 + (y-3)^2 = (\frac{6}{5})^2$. Show that $C_1 \cap C_2 = \emptyset$.

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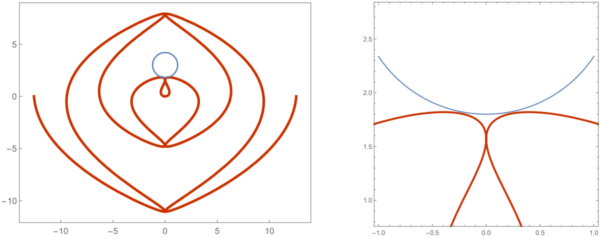
- This can be very hard for other methods (a numerical method for instance) because C_1 and C_2 are very close to each other, see below.



(a) Curve C_1 for $t \in (-4\pi, 4\pi)$ in red and circle C_2 in blue;

(b) Zoom in of the left graph, they nearly intersect

Figure 3: Curve C_1 and C_2.

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- How can we decide if this is true or false?

Problem Statement

- **Definition (Mixed Trigonometric-Polynomial)**

  A (rational coefficient, univariate) **mixed trigonometric-polynomial** (abbrev. MTP) is a function $f : \mathbb{R} \to \mathbb{R}$ that can be written as a rational coefficient polynomial in x, $\sin x$ and $\cos x$. That is $f(x) = \varphi(x, \sin x, \cos x)$ for some $\varphi \in \mathbb{Q}[x, y, z]$.

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- **The Decision Problem**

  The atoms in the first-order theory of MTPs are of the form $f \triangleright 0$, where f is an MTP, $\triangleright \in \{<, =, >, \neq, \leq, \geq\}$. A quantifier-free sentence $\Psi(x)$ is a boolean combination of these atoms. A closed sentence is a quantified sentence $(\forall x)\Psi(x)$ or $(\exists x)\Psi(x)$.
  Is there **an algorithm to decide** whether a closed sentence $\Phi(x) = (Qx)\Psi(x)$ $(Q \in \{\forall, \exists\})$ is true or false?

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- Richardson (1969) proved that the general theory containing composition of polynomials, $\exp x$ and $\sin x$ with two extra constants $\log 2$ and π is **undecidable**.
- The undecidability result was later improved by Caviness (1970) and Wang (1974).

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- Laczkovich (2003) showed even more, he proved that the ring generated by functions x, $\sin(x^n)$ and $\sin(x\sin(x^n))\,(n=1,2,\ldots)$ is undecidable over the reals.

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- Recently, Chen et al. (2022) gave an algorithm to isolate the real roots of MTP.

The Decidability of the Univariate MTP Theory

Recall that we are concerned about MTPs. In our paper, we prove that the first-order theory of univariate MTP is, **surprisingly unconditionally decidable**.

Theorem (ISSAC '23, Chen & Xia, Cor. 4.3)

The first-order theory of univariate MTPs is decidable.

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$$x \mapsto 2 \cdot \tfrac{x}{2}, \ \sin x \mapsto \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \ \cos x \mapsto \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$

We call $\varphi(f)$ **the canonical form** of an MTP f.

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- E.g. $\sin^2 x + \cos^2 x = \frac{(2\tan \frac{x}{2})^2 + (1-\tan^2 \frac{x}{2})^2}{(1+\tan^2 \frac{x}{2})^2} = \frac{(1+\tan^2 \frac{x}{2})^2}{(1+\tan^2 \frac{x}{2})^2} = 1.$
- It is always safe to consider only the numerator of the canonical form, because the denominator $(1+\tan^2 \frac{x}{2})^d > 0$.

Reduction II

- The next reduction comes from some basic **real algebraic geometry**.

Theorem

Given a bivariate polynomial $g \in \mathbb{R}[x, y]$ and g_0 is the square-free part of it, let M be an upper bound for all real roots $\mathtt{lc}_y(g_0) = 0$ and $\mathtt{discr}_y(g_0) = 0$. Then there is an integer $r \geq 0$ such that for all $x_0 > M$: $g(x_0, Y) = 0$ has exactly r distinct real roots in Y. Let $y_i(x)$ be the i-th root of $g(x, Y) = 0$ ($1 \leq i \leq r$, roots are numbered in the ascending order), then $y_i(x)$ is an algebraic function.

Reduction II (continued)

- Roughly speaking, this theorem says that the number of real roots of a 1-parameter and 1-unknown equation is a constant when the parameter is sufficiently large. Also the real roots are algebraic functions with respect to the parameter.

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- This allows us to reduce solving $g(x, \tan x) = 0$ to solving $\left\{\begin{array}{rcl} g(x,y) & = & 0 \\ y & = & \tan x \end{array}\right.$, then to solving $y_i(x) = \tan x$ for sufficiently large x.

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- This is, in fact equivalent to finding the intersections of the graphs of $y_i(x)$ and $\tan x$.
- Some pictures could be helpful.

Reduction II (explanation)

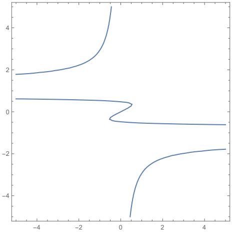

Figure 4: Locus of $g(x,y) = 10xy^8 - 40xy^6 + 60xy^4 - 40xy^2 + 10x + 20y^7 - 28y^5 + 92y^3 - 20y = 0$

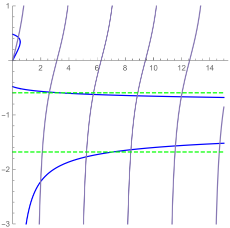

Figure 5: Local picture of $g(x,y) = 0$ and $y = \tan x$

In this example, $g(x,y) = 0$ gives two algebraic functions $y_1(x)$ and $y_2(x)$ that satisfy $g(x, y_i(x)) = 0$ for large x, and the solutions to $g(x, \tan x) = 0$ correspond to the intersections of $y = y_i(x)$ and $y = \tan x$.

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- If this iteration converges, then the limit is a fixed point.

Reduction III - Contraction Mapping Associated with an Alg. Func.

- In our case, finding roots of $\tan x = y_i(x)$ in $(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$ is equivalent to finding fixed points of $x = \arctan y_i(x) + k\pi$.

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- This leads to the following definition:

Definition (ISSAC '23, Chen & Xia, Def. 3.1)

Let k be a positive integer and let f be an algebraic function. The **k-th contraction mapping $T_{f,k}$ associated with f** is defined to be:

$$T_{f,k}: \begin{array}{rcl} [k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}] & \to & [k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}] \\ x & \mapsto & \arctan(f(x)) + k\pi. \end{array}$$

We use notation $T_{f,k}^{(m)}$ to denote the m-times composition of $T_{f,k}$ with itself:

$$T_{f,k}^{(m)} := \overbrace{T_{f,k} \circ \cdots \circ T_{f,k}}^{\text{m fold}}.$$

When f is clear, it can be omitted from the notation: T_k and $T_k^{(m)}$.

Reduction III - Contraction Mapping Associated with an Alg. Func. (cont'd)

Here are some examples showing what our new definition is.

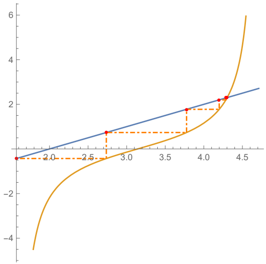

Figure 6: $f = x - 2$, $k = 1$

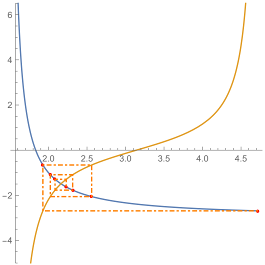

Figure 7: $f = \frac{1}{x-3/2} - 3$, $k = 1$

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- Recall the Banach Fixed Point Theorem:
- A contraction mapping defined on a complete metric space admits a unique fixed point.
- Observe that $T'_{f,k}(x) = \frac{f'(x)}{1+f^2(x)} \to 0 \, (x \to +\infty)$.
- So T_k are indeed contraction mappings and we can prove our main theorem:

Theorem (ISSAC '23, Chen & Xia, Thm. 3.3)

Let f be an algebraic function and T_k is the k-th contraction mapping associated with f. Then there exists an (effective) integer k_+ such that for all $k > k_+$: T_k has a unique fixed point $r \in (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$, and for any initial value $x_0 \in [k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}]$, the iteration:

$$x_1 = T_k(x_0), \; x_2 = T_k(x_1), \; x_3 = T_k(x_2), \; \cdots$$

converges to r.

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- **Corollary (ISSAC '23, Chen & Xia, Cor. 3.4)**

  Suppose $g \in \mathbb{Q}[x, t]$, then there are effective integers c_+, k_+ (resp. c_-, k_-) such that there are exactly c_+ (resp. c_-) roots of $g(x, \tan x) = 0$ in $(k\pi - \frac{1}{2}\pi, k\pi + \frac{1}{2}\pi)$ for all $k > k_+$ (resp. $k < k_-$).

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- The secret of effectiveness comes from the fact that $|T'_{f,k}(x)| < \tau$ is captured by purely algebraic conditions

$$\mathtt{res}_t\left(\frac{\partial g_i}{\partial x} \pm \tau(t^2+1)\frac{\partial g_i}{\partial t}, g_i\right)(x) = 0,$$

where g_i are the irreducible factors of g.

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- **Theorem (ISSAC '23, Chen & Xia, Thm. 4.2)**

  Let $\Phi(x)$ be a quantifier-free formula in the theory of univariate MTPs. There are effective $K_-, K_+ \in \mathbb{Z}$ such that $\Phi(x)$ is true for all $x \in \mathbb{R}$ if and only if $\Phi(x)$ is true for all $x \in (2K_-\pi - \pi, 2K_+\pi + \pi)$.

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- In other words, this theorem reduces a decision problem over the reals to an equivalent problem over a bounded interval. The latter problem is already shown to be decidable by McCallum and Weispfenning (2012). The decidability result is established now.

1. Problem
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Package `UnivariateMTPDecisionV2`

We implement the decision algorithm with Mathematica 12. Our package `UnivariateMTPDecisionV2` is available at:

https://github.com/xiaxueqaq/MTP-decision.

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```
In[2]:= Timing[DecideMTP[b1 == 0, {b1}, {-36/25 + t^2 Cos[t]^4 + (-3 + t Sin[t])^2}, t, False]]
        计算时间                                                                     假

Out[2]=
        {0.1875, False}
```

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- The algorithm confirms that $C_1 \cap C_2 = \emptyset$ in 0.1875s.
  ```

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Timing[DecideMTP[((b1 == 0) ∧ (b2 > 0)) ⇒ (b3 > 0),
  {b1, b2, b3}, {x^2 Cos[x] - Sin[x], x, x - 1}, x]]
```

计算时间 余弦 正弦

Out[3]=

```
{0.078125, True}
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                            余弦        正弦

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```

- The algorithm proves it in 0.08s.

Experiments

In our paper, to test the algorithm, we choose 6 examples that can be formulated in the first-order language of univariate MTPs, including

- theorem proving,

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Our algorithm succeeds in proving all sentences, and is shown to be very efficient. The results are briefly summarized in the next page.

Experiments (continued)

The experiment was conducted on a laptop that runs Windows 10 with an Intel Core i7-10750H@2.60GHz (6 Cores, 12 Threads) processor and 8 GB of RAM.

Examples	1	2	3	4	5	6
Time (s)	0.047	0.172	0.031	0.219	2.438	0.031
Quantifier	$\forall$	$\forall$	$\forall$	$\exists$	$\exists$	$\forall$
Number of MTPs	3	3	1	1	4	2
Degree*	(4,2)	(6,4)	(0,2)	(2,8)	(0,16)	(6,5)

* - The degree in x and $\tan x$ of the canonical form of the product of MTPs.

Table 1: A brief summary of the experiment

Conclusion

- In our paper, we prove that the first-order theory of univariate MTP is decidable, and the result does not rely on any unproven conjecture.
- Also, we implement the decision algorithm with Mathematica 12. Some experiments show that our algorithm is quite efficient.

Future Work

- Complexity?
- Is it possible to generalize to $\mathbb{C}$?
- Is it possible to generalize to $\mathbb{Q}[x, \sin\alpha_1 x, \cos\alpha_1 x, \ldots, \sin\alpha_n x, \cos\alpha_n x]$, where α_i are real algebraic numbers?
- ...

Thank you!

You are more than welcome to give any suggestion!