Reduction of Transcendental Decision Problems over the Reals

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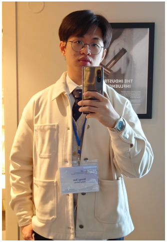
Authors



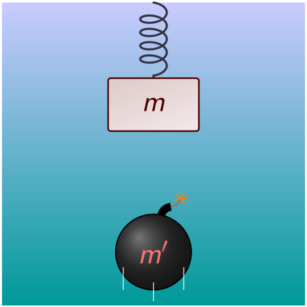
Author 1: Rizeng Chen (pronounced as "ReeTseng Chern")

Author 2: Bican Xia (pronounced as "BeeChan Shia")

I am Rizeng Chen, the people on the left and I am currently a third-year PhD candidate under the supervision of Prof. Xia.

Problems with Transcendental Constraints

A mass is attached to a spring in the water. A bomb is thrown beneath the mass. The bomb will explode when it hits the mass!



Question: Will the mass and the bomb collide at some time $t > 0$?

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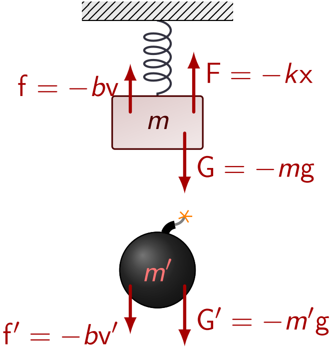

Figure 1: Free Body Diagram

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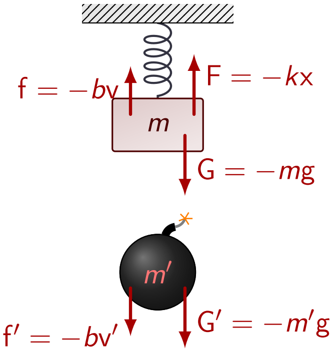

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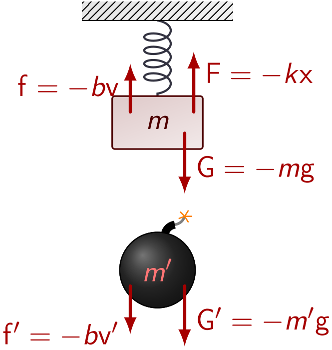

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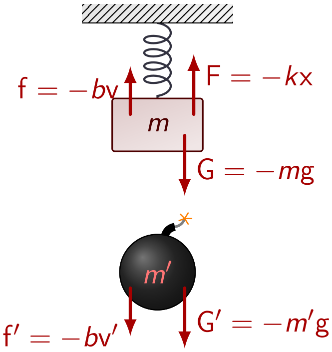

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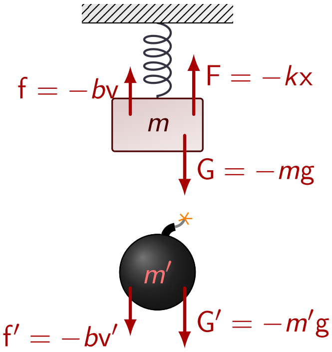

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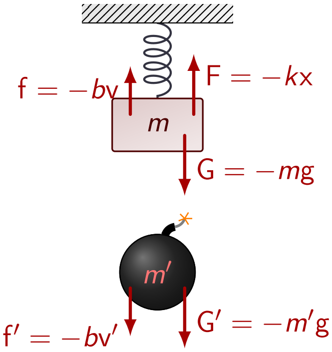

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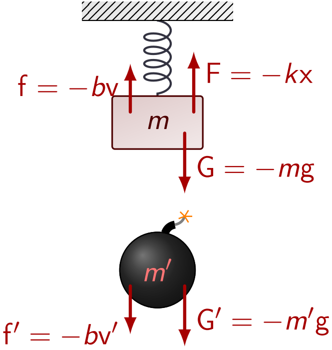

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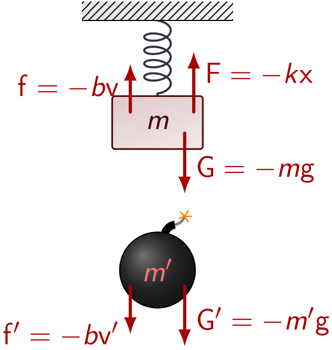

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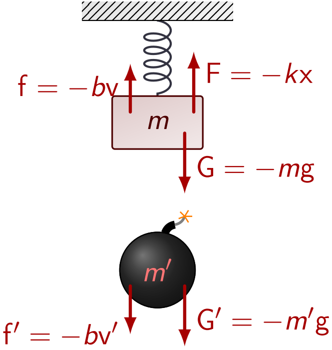

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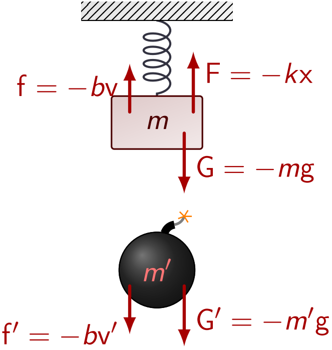

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- Suppose $m = m' = 1\mathrm{kg}$, $k = 10\mathrm{N/m}$, $b = 2\mathrm{N}\cdot\mathrm{s/m}$ and $g = 10\mathrm{m/s}^2$.
- The initial positions are $x(0) = 0\mathrm{m}$, $y(0) = -5\mathrm{m}$, and the initial velocities are $\dot{x}(0) = -12\mathrm{m/s}$ and $\dot{y}(0) = 9\mathrm{m/s}$.

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- So the kinetic equations are

$$\begin{cases} \ddot{x} &= -2\dot{x} - 10x - 10 \\ \dot{x}(0) &= -12 \\ x(0) &= 0 \end{cases}, \begin{cases} \ddot{y} &= -2\dot{y} - 10 \\ \dot{y}(0) &= 9 \\ y(0) &= -5 \end{cases}.$$

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- Solving the ODE:

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- Transcendental constraints naturally arise in the real world.
- How do we handle them?

Polynomials are easy...

- Tarski (1951) showed that the first-order theory over a real closed field is decidable (polynomial equations and inequalities).

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- **Schanuel's Conjecture (SC)**

  Suppose $z_1, \ldots, z_n \in \mathbb{C}$ are linear independent over $\mathbb{Q}$, then

  $$\operatorname{tr.deg} \mathbb{Q}(z_1, \ldots, z_n, \mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_n}) \geq n$$

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- Achatz, McCallum and Weispfenning (2008) presented an algorithm to decide polynomial-exponential problems (later generalized to $\ln x$ and $\arctan x$).
- At the same time, Strzeboński (2008) studied the real root isolation of exp-log functions (then extended to tame elementary functions and exp-log-arctan functions).

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- It is shown in Chen and Xia (2023) that the theory of univariate mixed trigonometric-polynomials (MTP) is **surprisingly decidable**.

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- Hence, a reduction from the unbounded case to the bounded case should be favorable.

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  Let $f_i \in S[\sin x, \cos x]$ for $i = 1, \ldots, s$, then there are effective bounds $N, M \in \mathbb{R}$ such that any quantifier-free formula $\varphi(x)$ whose atoms are of the form $f_i \triangleright 0$ is true for all $x \in \mathbb{R}$ if and only if $\varphi(x)$ is true for all $x \in [N, M]$.

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- Our ISSAC'23 result can be directly derived from this by setting $S = \mathbb{Q}[x]$.

Applications of the Reduction Theorem

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As a corollary, the reachability problem of a linear differential system $\dot{\mathrm{x}}(t) = A\mathrm{x}(t)$ is decidable, if the imaginary part of eigenvalues of A spans a 1-dimensional space over $\mathbb{Q}$.

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  So it suffices to consider $S[\tan \frac{x}{2}]$ instead of $S[\sin x, \cos x]$.
- The singularity of $\tan \frac{x}{2}$ at $x = (2k+1)\pi$ $(k \in \mathbb{Z})$ can be treated separately.

Essential Ingredient From Real Algebraic Geometry

Collins (1975) proposed Cylindrical Algebraic Decomposition (CAD). The key theorem in the paper is:

Theorem (Col75, Thm. 1)

Let $f(\vec{x}; y)$ be a parametric univariate polynomial and let C be a connected parameter region.
Suppose $\mathtt{LC}_y(f) \neq 0$ for all $\vec{x} \in C$ and the number of distinct complex roots of $f(\vec{x}; y)$ is invariant for all $\vec{x} \in C$.
Then f is delineable over C, i.e. the real roots of f are continuous functions in the parameters.

He counted the complex roots by his subresultant theory.

Essential Ingredient From Real Algebraic Geometry (cont'd)

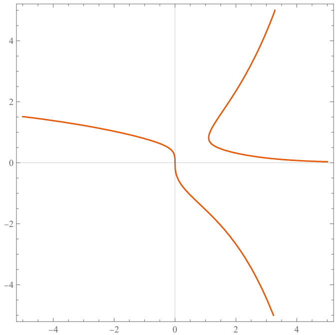

Figure 2: The locus of $f(x, y) = y^3 - (e^x - 1)y + x$

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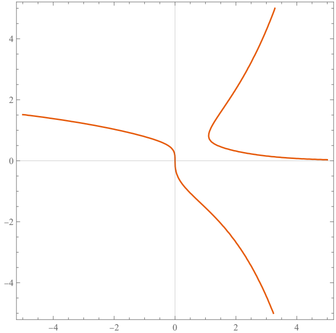

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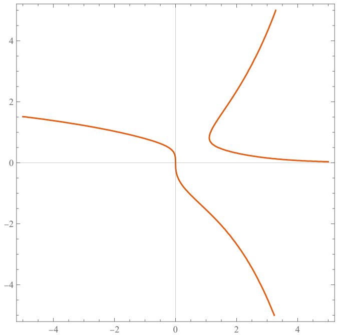

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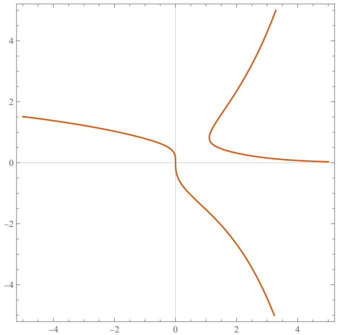

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  3. $\Delta_2 = 3$.

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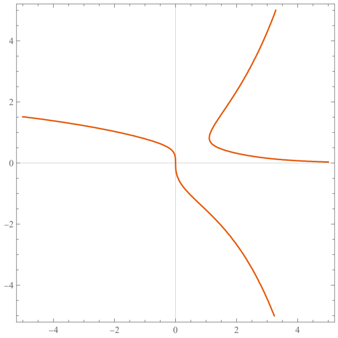

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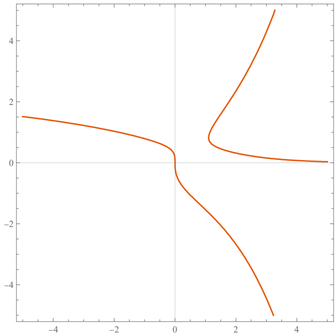

Figure 2: The locus of $f(x, y) = y^3 - (\mathrm{e}^x - 1)y + x$

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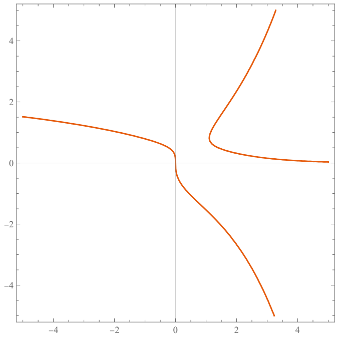

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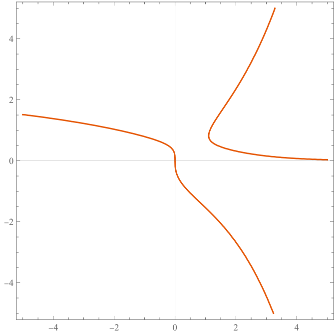

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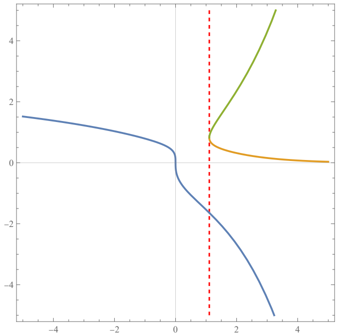

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- For all $x_0 > 1.105\ldots$, $\Delta_0 \neq 0$, thus the real roots are continuous functions $y_1(x)$, $y_2(x)$ and $y_3(x)$.

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- e.g. $\tilde{\varphi}(x, y) =$ "$y^3 - (\mathrm{e}^x - 1)y + x < 0$".
- Note that the subdiscriminants $\Delta_0, \Delta_1, \ldots$ of $h \in S[y]$ is always in S, so they have finitely many real roots if they are not identically zero.

The Proof for the Reduction Theorem (cont'd)

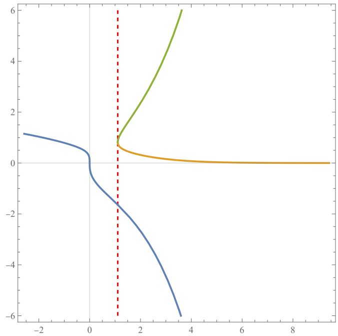

Figure 3: $h(x, y)$ is delineable over $I = (1.105\ldots, +\infty)$

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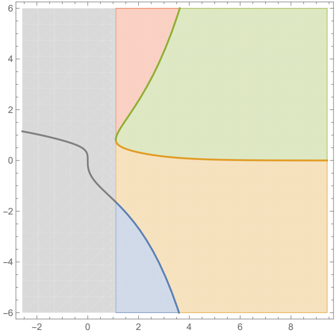

Figure 3: $I \times \mathbb{R}$ consists of cylindrically stacked cells

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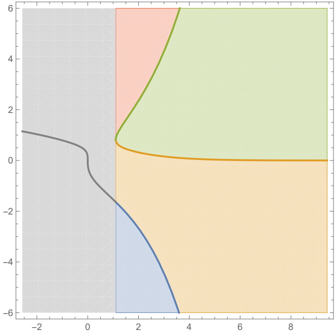

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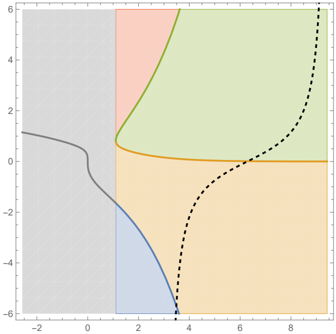

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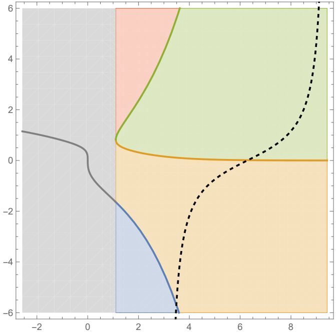

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- It is easy to observe that the graph of $y = \tan \frac{x}{2}$ intersects with each cell in each period.
- Hence, if $\tilde{\varphi}(x, y)$ holds in cell C, $\varphi = \tilde{\varphi}(x, \tan \frac{x}{2})$ is also satisfiable in each period $(2k\pi - \pi, 2k\pi + \pi) \subseteq I$. It suffices to look at one of them.

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- Hence, our result in the univariate case is not very far from being optimal.

Package TranscendentalProblems

We implement the reduction algorithm with Mathematica 13. Our package TranscendentalProblems is available at:

https://github.com/xiaxueqaq/TranscendentalProblems.

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Experiments

We report some experimental data here.

Examples	1	2	3	4	5	6
Time (s)	1.531	0.969	1.797	3.562	1.891	1.438
Examples	7	8	9	10	11	12
Time (s)	0.125	0.031	0.188	0.078	8.406	2.109

Table 1: Running Time

Thank you!

You are more than welcome to give any suggestion!